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Journal of Geometry and Physics 41 (2002) 359–375

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Spacelike hypersurfaces with constant higher order mean curvature in Minkowski space–time

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Received 11 June 2000; accepted 13 September 2001

Abstract

In this paper, we develop a series of general integral formulae for compact spacelike hypersurfaces with hyperplanar boundary in the $(n+1)$-dimensional Minkowski space–time $\mathbb{L}^{n+1}$. As an application of them, we prove that the only compact spacelike hypersurfaces in $\mathbb{L}^{n+1}$ having constant higher order mean curvature and spherical boundary are the hyperplanar balls (with zero higher order mean curvature) and the hyperbolic caps (with nonzero constant higher order mean curvature). This extends previous results obtained by the first author, jointly with Pastor, for the case of constant mean curvature [J. Geom. Phys. 28 (1998) 85] and the case of constant scalar curvature [Ann. Global Anal. Geom. 18 (2000) 75]. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 53C42; 53B30; 53C50

Keywords: Minkowski space–time; Spacelike hypersurfaces; Higher order mean curvature; Flux formula; Hyperbolic caps

1. Introduction

The study of spacelike hypersurfaces in Lorentzian space–times has been of increasing interest in recent years from both physical and mathematical points of view. From a physical point of view, such interest is motivated by their role in different problems of general relativity. For instance, Lichnerowicz [14] showed that zero mean curvature spacelike hypersurfaces are convenient as initial data for solving the Cauchy problem of the Einstein equations. We also refer to [8,12,15,20] and references therein for other reasons justifying that interest.

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PII: S0393-0440(01)00069-9

From a mathematical point of view, spacelike hypersurfaces are also interesting because of their nice Bernstein-type properties. Recall that the Bernstein problem for maximal hypersurfaces in the Minkowski space–time $\mathbb{L}^{n+1}$ was first introduced by Calabi [10], where he found that, for $n \leq 4$, the only entire solutions to the maximal hypersurface equation in $\mathbb{L}^{n+1}$ are affine functions. This result was extended to the general n-dimensional case by Cheng and Yau [11], who proved that the only complete maximal hypersurfaces in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. On the other hand, Aiyama [1] and Xin [21] simultaneous and independently characterized the spacelike hyperplanes as the only complete spacelike hypersurfaces with constant mean curvature in $\mathbb{L}^{n+1}$ whose Gauss map image is bounded in the hyperbolic image (see also [17] for a weaker first version of this result by Palmer).

In a series of recent papers, the first author, jointly with Pastor, studied the geometry of compact spacelike hypersurfaces (necessarily with non-empty boundary) in the Minkowski space–time. In particular, in [5], they showed that the only such hypersurfaces having constant mean curvature and round spherical boundary are the hyperplanar balls and the hyperbolic caps [5] (see also [3] for a first two-dimensional version of this result). As for the case of the scalar curvature, they characterized the hyperbolic caps in $\mathbb{L}^{n+1}$ as the only compact spacelike hypersurfaces in the Minkowski space–time with nonzero constant scalar curvature and spherical boundary [6].

Their approach to obtain those results in [5,6] was based on the use of certain integral formulae for the case, where either the mean curvature or the scalar curvature is constant. In this paper, we will develop a series of general integral formulae for compact spacelike hypersurfaces with hyperplanar boundary in $\mathbb{L}^{n+1}$, for the case where a higher order rth mean curvature is constant, $1 \leq r \leq n$. Let us recall that the higher order mean curvatures H_r of a hypersurface are the natural generalization of mean and scalar curvature. Indeed, H_1 is simply the (extrinsic) mean curvature of the hypersurface and H_2 is, up to a constant, its (intrinsic) scalar curvature (for details, see Section 2). In particular, we will derive a flux formula (see Proposition 1 and formula (23)) which extends to the general case previous flux formulae given in [5] when $r = 1$ and in [6] when $r = 2$ (we also refer to the recent paper by Bahn and Hong [7] for another interesting flux-type formula for spacelike hypersurfaces in $\mathbb{L}^{n+1}$, with interesting applications to isoperimetric and some other geometric inequalities).

As a first application of this new flux formula, we are able to extend and generalize the characterization theorem given in [6] for the case of scalar curvature to the case of any intrinsic curvature. Specifically, we will obtain the following uniqueness result.

Theorem 1. *The only compact spacelike hypersurfaces in the Minkowski space–time with constant intrinsic higher order mean curvature H_r ($2 \leq r \leq n$, r even) and spherical boundary are the hyperplanar balls (with $H_r = 0$) and the hyperbolic caps (with positive constant H_r).*

Besides, we also develop a new integral inequality (see Proposition 2) which extends an inequality given in [5] when $r = 1$. Using this inequality, we are also able to extend our uniqueness theorem to the case of any extrinsic curvature.

Theorem 2. *The only compact spacelike hypersurfaces in the Minkowski space–time with constant extrinsic higher order mean curvature H_r ($1 \leq r \leq n$, r odd) and spherical* *boundary are the hyperplanar balls (with* $H_r = 0$) *and the hyperbolic caps (with nonzero constant* H_r).

2. Preliminaries

Let $\mathbb{L}^{n+1}$ denote the $(n+1)$-dimensional Minkowski space–time, that is, the real vector space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric

$$\langle,\rangle = -dx_0^2 + dx_1^2 + \cdots + dx_n^2,$$

where $(x_0, x_1, \ldots, x_n)$ are the canonical coordinates in $\mathbb{R}^{n+1}$. A smooth immersion $x : M^n \to \mathbb{L}^{n+1}$ of a n-dimensional connected manifold M is said to be a *spacelike hypersurface* if the induced metric via x is a Riemannian metric on M, which, as usual, is also denoted by $\langle,\rangle$. Observe that $(1, 0, \ldots, 0)$ is a unit timelike vector field globally defined on $\mathbb{L}^{n+1}$, which determines a time-orientation on $\mathbb{L}^{n+1}$. Thus, we can choose a unique unit normal vector field N on M which is a future-directed timelike vector in $\mathbb{L}^{n+1}$, and hence we may assume that M is oriented by N.

In order to set up the notation to be used later, let us denote by ∇^0 and ∇ the Levi–Civita connections of $\mathbb{L}^{n+1}$ and M, respectively. Then the Gauss and Weingarten formulae for M in $\mathbb{L}^{n+1}$ are written, respectively, as

$$\nabla^0_X Y = \nabla_X Y - \langle AX, Y\rangle N, \tag{1}$$

and

$$A(X) = -\nabla^0_X N, \tag{2}$$

for all tangent vector fields $X, Y \in \mathcal{X}(M)$, where A stands for the shape operator with respect to N.

Associated to the shape operator of M there are n algebraic invariants, which are the elementary symmetric functions of its principal curvatures $k_1, \ldots, k_n$ given by

$$S_r = S_r(k_1, \ldots, k_n) = \sum_{i_1<\cdots<i_r} k_{i_1} \ldots k_{i_r}, \quad 1 \le r \le n.$$

Following [2], we define the *'r'th mean curvature* H_r of the spacelike hypersurface by

$$\binom{n}{r} H_r = (-1)^r S_r(k_1, \ldots, k_n) = S_r(-k_1, \ldots, -k_n), \quad 1 \le r \le n.$$

Observe that when $r = 1$, $H_1 = -(1/n)\,\mathrm{tr}(A) = H$ is the mean curvature of M. The choice of the sign $(-1)^r$ in the above definition of H_r is motivated by the fact that in that case the mean curvature vector is given by $\mathbf{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in M$ if and only if $\mathbf{H}(p)$ is in the time-orientation determined by $N(p)$. On the other hand, when $r = n$, $H_n = (-1)^n \det(A)$ defines the Gauss–Kronecker curvature of the spacelike hypersurface, and for $r = 2$, H_2 is, up to a constant, the scalar curvature S of M, since $S = -n(n-1)H_2$. In general, it follows from Gauss equation of the hypersurface that when r is odd H_r is extrinsic (and its sign depends on the chosen orientation), while when r is even H_r is an intrinsic geometric quantity.

Throughout this work, we will deal with *compact* spacelike hypersurfaces immersed in $\mathbb{L}^{n+1}$. Let us remark that there exists no closed (compact without boundary) spacelike hypersurfaces in $\mathbb{L}^{n+1}$. To see this, let $a \in \mathbb{L}^{n+1}$ be a fixed arbitrary vector, and consider the height function $\langle a, x\rangle$ defined on the spacelike hypersurface M. The gradient on M of $\langle a, x\rangle$ is

$$\nabla\langle a, x\rangle = a^{\mathrm{T}} = a + \langle a, N\rangle N,$$

where $a^{\mathrm{T}} \in \mathcal{X}(M)$ is tangent to M. In fact, it is easy to see that $a = a^{\mathrm{T}} - \langle a, N\rangle N$, and for any $X \in \mathcal{X}(M)$, we have

$$\langle\nabla\langle a, x\rangle, X\rangle = X\langle a, x\rangle = \langle\nabla^0_X a, x\rangle + \langle a, \nabla^0_X x\rangle = \langle a, X\rangle.$$

From this, we conclude that

$$|\nabla\langle a, x\rangle|^2 = \langle a, a\rangle + \langle a, N\rangle^2 \geq \langle a, a\rangle.$$

In particular, when a is spacelike the height function has no critical points in M, so that M cannot be closed. Therefore, every compact spacelike hypersurface M necessarily has non-empty boundary ∂M. As usual, if Σ is an $(n-1)$-dimensional closed submanifold in $\mathbb{L}^{n+1}$, a spacelike hypersurface $x : M \to \mathbb{L}^{n+1}$ is said to be a hypersurface *with boundary* Σ if the restriction of the immersion x to the boundary ∂M is a diffeomorphism onto Σ.

In what follows, $x : M \to \mathbb{L}^{n+1}$ will be a compact spacelike hypersurface with boundary ∂M, and we will consider M oriented by a unit timelike normal vector field N. The orientation of M induces a natural orientation on ∂M as follows: a basis $\{v_1, \dots, v_{n-1}\}$ for $T_p(\partial M)$ is positively oriented if and only if $\{u, v_1, \dots, v_{n-1}\}$ is a positively oriented basis for T_pM, whenever $u \in T_pM$ is outward pointing. We will denote by ν the outward pointing unit conormal vector field along ∂M.

A specially interesting case occurs when the boundary $\Sigma = x(\partial M)$ is contained in a fixed hyperplane Π of $\mathbb{L}^{n+1}$. We will refer to it saying that M has *hyperplanar* boundary. Since Σ is closed, it follows that the hyperplane Π is spacelike. We can assume without loss of generality that Π passes through the origin and $\Pi = a^{\perp}$, for a unit timelike vector $a \in \mathbb{L}^{n+1}$ in the same time-orientation as N. The following lemma and its corollary below will be essential later on.

Lemma 1 (Existence of an elliptic point). *Let $x : M \to \mathbb{L}^{n+1}$ be a spacelike immersion of a compact hypersurface with hyperplanar boundary $\Sigma = x(\partial M)$, and assume that Σ is contained in a hyperplane $\Pi = a^{\perp}$, 'a' being a unit timelike vector in the same time-orientation as N. Then there exists a point $p_0 \in$ int(M) where (after an appropriate choice of orientation on M) all the principal curvatures are negative, unless the hypersurface is entirely contained in the hyperplane Π.*

Proof. Let us assume that the hypersurface is not entirely contained in the hyperplane Π. Then, there exists a point $p \in \mathrm{int}(M)$, where $\langle x(p), a\rangle \neq 0$. We may assume without loss of generality that $\langle x(p), a\rangle < 0$. Otherwise, just replace a by $-a$ and change the orientation on M (recall that we are assuming that the timelike vectors a and N are in the same time-cone). Let $b \in \Pi$ be the projection of $x(p)$ onto Π, $b = x(p) + \langle x(p), a\rangle a$, and let us denote by $\mathbb{H}^n_+(b, \rho)$ the connected component of $\langle q - b, q - b\rangle = -\rho^2$ which satisfies $\langle q - b, a\rangle = \langle q, a\rangle < 0$, that is,

$$\mathbb{H}^n_+(b, \rho) = \{q \in \mathbb{L}^{n+1} : \langle q - b, q - b\rangle = -\rho^2, \langle q, a\rangle < 0\}.$$

Since M is compact, $\mathbb{H}^n_+(b, \rho)$ does not intersect $x(M)$ for large ρ. As ρ decreases, there exists an interior point $p_0 \in \operatorname{int}(M)$, where $\mathbb{H}^n_+(b, \rho)$ touches $x(M)$. Such a point p_0 satisfies $\langle x(p_0), a\rangle < 0$ and it is a local maximum point for the function $u = \langle x - b, x - b\rangle$ defined on M. In particular, its gradient vanishes at p_0, $\nabla u(p_0) = 0$, and its Hessian satisfies

$$\nabla^2 u_{p_0}(v, v) \le 0$$

for every tangent vector $v \in T_{p_0}M$. It is easy to see that $\nabla u = 2(x - b)^{\mathrm{T}}$, where

$$(x - b)^{\mathrm{T}} = x - b + \langle x - b, N\rangle N$$

is tangent to M. Taking covariant derivative here and using Gauss (Eq. (1)) and Weingarten formulae (Eq. (2)), we also see that

$$\nabla^2 u(X, Y) = 2\langle X, Y\rangle - 2\langle x - b, N\rangle\langle AX, Y\rangle, \quad \text{for } X, Y \in \mathcal{X}(M).$$

From $\nabla u(p_0) = 0$, it follows that

$$\langle x(p_0) - b, N(p_0)\rangle = -\frac{\langle x(p_0), a\rangle}{\langle N(p_0), a\rangle} < 0. \tag{3}$$

Besides, let $\{e_1, \dots, e_n\}$ be a basis of principal directions at p_0 satisfying $A_{p_0}(e_i) = k_i(p_0)e_i$. Then

$$\nabla^2 u_{p_0}(e_i, e_i) = 2[1 - \langle x(p_0) - b, N(p_0)\rangle k_i(p_0)] \le 0,$$

which from Eq. (3) gives

$$k_i(p_0) \le \frac{1}{\langle x(p_0) - b, N(p_0)\rangle} < 0. \qquad \square$$

Corollary 1. *Let $x : M \to \mathbb{L}^{n+1}$ be a spacelike immersion of a compact hypersurface bounded by an $(n - 1)$-dimensional submanifold $\Sigma = x(\partial M)$, and assume that Σ is contained in a hyperplane $\Pi = a^{\perp}$, ‘a’ being a unit timelike vector in the same time-orientation as N. If the ‘r’th mean curvature H_r is constant, then either $H_r = 0$ and the hypersurface is just the hyperplanar domain in Π bounded by Σ, or (after an appropriate choice of orientation on M) H_r is positive and it holds that*

$$H_1 \ge H_2^{1/2} \ge \cdots \ge H_{r-1}^{1/(r-1)} \ge H_r^{1/r} > 0.$$

Besides, equality holds at any stage only at umbilical points.

Proof. Let us assume that the hypersurface is not the hyperplanar domain in Π bounded by Σ. Then, there exists a point $p_0 \in M$, where all the principal curvatures are negative. The result then follows from the proof of [16] (Lemma 1) (see also [9] (Section 12)), taking into account the sign convention in our definition of the higher order mean curvatures. □

3. A flux formula

In this section, we will derive a flux formula for compact spacelike hypersurfaces in the Minkowski space–time. In order to do that, we will introduce the corresponding Newton transformations $T_r : \mathcal{X}(M) \to \mathcal{X}(M)$ associated with the second fundamental form, which according to our definition of the rth mean curvatures are given by

$$T_r = \binom{n}{r} H_r I + \binom{n}{r-1} H_{r-1} A + \cdots + \binom{n}{1} H_1 A^{r-1} + A^r,$$

where I denotes the identity in $\mathcal{X}(M)$, or inductively,

$$T_0 = I \quad \text{and} \quad T_r = \binom{n}{r} H_r I + A T_{r-1}. \tag{4}$$

Observe that the characteristic polynomial of A can be written in terms of the H_r's as

$$\det(tI - A) = \sum_{r=0}^{n} \binom{n}{r} H_r t^{n-r}, \tag{5}$$

where $H_0 = 1$ by definition. By Cayley–Hamilton theorem, this implies that $T_n = 0$.

Let us remark that $T_r = (-1)^r P_r$, where P_r is the classical rth Newton transformation defined by Reilly in [18] (see also [19] for a more accesible modern treatment). Observe that the Newton transformations T_r are all self-adjoint operators which commute with the shape operator A. Besides, we have the following nice properties of T_r.

1. If $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M which diagonalizes A, $Ae_i = k_i e_i$, then it also diagonalizes each T_r, and $T_r e_i = \mu_{i,r} e_i$ with
$$\mu_{i,r} = (-1)^r S_r(k_1, \ldots, \hat{k}_i, \ldots, k_n) = (-1)^r \sum_{i_1 < \cdots < i_r, i_j \neq i} k_{i_1}, \ldots, k_{i_r}.$$
2. For each $1 \leq r \leq n-1$, $\mathrm{tr}(T_r) = (r+1)\binom{n}{r+1} H_r$, and
$$\mathrm{tr}(AT_r) = -(r+1)\binom{n}{r+1} H_{r+1}. \tag{6}$$
3. For every $X \in \mathcal{X}(M)$ and for each $1 \leq r \leq n-1$,
$$\mathrm{tr}(T_r \nabla_X A) = -\binom{n}{r+1} \langle \nabla H_{r+1}, X\rangle.$$

4. The Newton transformations T_r are divergence free, that is,

$$\operatorname{div}_M(T_r) = \operatorname{tr}(X \to (\nabla_X T_r)X) = 0. \tag{7}$$

Let $Y \in \mathcal{X}(\mathbb{L}^{n+1})$ be a Killing vector field on $\mathbb{L}^{n+1}$. We can write the restriction of Y to M as $Y = Y^{\mathrm{T}} - \langle Y, N\rangle N$, where $Y^{\mathrm{T}} \in \mathcal{X}(M)$ is tangent to M. Our objective is to compute $\operatorname{div}_M(T_r Y^{\mathrm{T}})$. Taking into account that $\nabla_X T_r$ is self-adjoint for any $X \in \mathcal{X}(M)$, an easy computation using Eq. (7) shows that

$$\operatorname{div}_M(T_r Y^{\mathrm{T}}) = \langle \operatorname{div}_M(T_r), Y\rangle + \sum_{i=1}^{n} \langle \nabla_{e_i} Y^{\mathrm{T}}, T_r e_i\rangle = \sum_{i=1}^{n} \langle \nabla_{e_i} Y^{\mathrm{T}}, T_r e_i\rangle, \tag{8}$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M. Let X be a tangent vector field to M. From Killing equation, we know that

$$\langle \nabla^0_{T_r X} Y, X\rangle + \langle \nabla^0_X Y, T_r X\rangle = 0,$$

which gives

$$\langle \nabla_{T_r X} Y^{\mathrm{T}}, X\rangle + \langle \nabla_X Y^{\mathrm{T}}, T_r X\rangle = -2\langle Y, N\rangle \langle A T_r X, X\rangle. \tag{9}$$

Computing in a local orthonormal frame on M that diagonalizes A, and hence T_r, we have by (1) above that

$$\langle \nabla_{T_r e_i} Y^{\mathrm{T}}, e_i\rangle = \mu_{i,r} \langle \nabla_{e_i} Y^{\mathrm{T}}, e_i\rangle = \langle \nabla_{e_i} Y^{\mathrm{T}}, T_r e_i\rangle,$$

so that from Eq. (9)

$$\langle \nabla_{e_i} Y^{\mathrm{T}}, T_r e_i\rangle = -\langle Y, N\rangle \langle A T_r e_i, e_i\rangle.$$

Therefore, we conclude from Eq. (8), using also Eq. (6), that

$$\operatorname{div}_M(T_r Y^{\mathrm{T}}) = -\langle Y, N\rangle \operatorname{tr}(A T_r) = (r+1)\binom{n}{r+1} H_{r+1}\langle Y, N\rangle. \tag{10}$$

Now integrating Eq. (10) on M, we have by the divergence theorem that

$$\oint_{\partial M} \langle T_r \nu, Y\rangle \, \mathrm{d}s = \int_M \operatorname{div}_M(T_r Y^{\mathrm{T}}) \, \mathrm{d}M = (r+1)\binom{n}{r+1} \int_M H_{r+1}\langle Y, N\rangle \, \mathrm{d}M,$$
$$\text{for every } 0 \le r \le n-1. \tag{11}$$

Here, $\mathrm{d}M$ stands for the n-dimensional volume element of M with respect to the induced metric and the chosen orientation, and $\mathrm{d}s$ is the induced $(n-1)$-dimensional volume element on ∂M.

On the other hand, consider D is a compact orientable hypersurface of $\mathbb{L}^{n+1}$ with smooth boundary that satisfies $\partial D = \partial M$. Assume that $M \cup D$ is an oriented n-cycle of $\mathbb{L}^{n+1}$. Let n_D be the unit normal that orients D. From the Alexander duality theorem, we have that $M \cup D = \partial \Omega$, where Ω is a compact oriented domain immersed in $\mathbb{L}^{n+1}$. From

Killing equation again, $\langle \nabla^0_Z Y, Z\rangle = 0$ for all $Z \in \mathcal{X}(\mathbb{L}^{n+1})$, from which it follows that $\mathrm{div}_{\mathbb{L}^{n+1}} Y = 0$. Therefore, from divergence theorem, we have

$$\int_D \langle Y, n_D\rangle \,\mathrm{d}A = -\int_M \langle Y, N\rangle \,\mathrm{d}M. \tag{12}$$

Here, $\mathrm{d}A$ stands for the n-dimensional volume element of D with respect to the induced metric and the chosen orientation.

Therefore, in the case where the hypersurface has some higher order mean curvature H_r constant, we obtain from Eqs. (11) and (12), the following *flux formula* for immersed spacelike hypersurfaces with constant higher order mean curvature in the Minkowski space–time. The corresponding formula for hypersurfaces in Euclidean space can be found in [19]; see also [4] for the others Riemannian space forms.

Theorem 3. *Let $x : M \to \mathbb{L}^{n+1}$ be a spacelike immersion of a compact hypersurface with boundary ∂M, and let 'D' be a compact orientable hypersurface of $\mathbb{L}^{n+1}$ with smooth boundary that satisfies $\partial D = \partial M$. Let n_D be the unit normal that orients D. If the 'r'th mean curvature H_r is constant, $1 \le r \le n$, then for any Killing vector field Y on $\mathbb{L}^{n+1}$*

$$\oint_{\partial M} \langle T_{r-1}\nu, Y\rangle \,\mathrm{d}s = -r\binom{n}{r} H_r \int_D \langle Y, n_D\rangle \,\mathrm{d}A. \tag{13}$$

Here, ν is the outward pointing conormal to M along ∂M.

Let us assume from now on that M has hyperplanar boundary. We may assume without loss of generality that the boundary is contained in a hyperplane Π which passes through the origin, and $\Pi = a^\perp$, for a unit timelike vector $a \in \mathbb{L}^{n+1}$ in the same time-orientation as N. In this situation, by considering the constant Killing field $Y = a$, we have that $n_D = -a$, and hence we know from Eq. (12) that

$$\mathrm{vol}(D) = -\int_M \langle a, N\rangle \,\mathrm{d}M. \tag{14}$$

This allows us to state the following result, which generalizes Proposition 2 in [5].

Proposition 1. *Let $x : M \to \mathbb{L}^{n+1}$ be a spacelike immersion of a compact hypersurface bounded by an $(n-1)$-dimensional embedded submanifold $\Sigma = x(\partial M)$, and assume that Σ is contained in a hyperplane Π of $\mathbb{L}^{n+1}$. Let 'a' be the unit timelike vector in $\mathbb{L}^{n+1}$ such that $\Pi = a^\perp$ which is the same time-orientation as N. If the 'r'th mean curvatureH_r is constant, then the $(r-1)$-flux*

$$\oint_{\partial M} \langle T_{r-1}\nu, a\rangle \,\mathrm{d}s$$

does not depend on the hypersurface, but only on the value of H_r and Σ. Actually,

$$\oint_{\partial M} \langle T_{r-1}\nu, a\rangle \,\mathrm{d}s = -r\binom{n}{r} H_r \,\mathrm{vol}(D), \tag{15}$$

where D is the domain in Π bounded by Σ.

4. An essential auxiliary lemma

In this section, our aim is to compute the term $\langle T_r\nu, a\rangle$ on the boundary of M in terms of the curvatures of the boundary. To do that, let $\{e_1, \ldots, e_{n-1}\}$ be a (locally defined) positively oriented frame field along ∂M. Since $\langle a, x\rangle = 0$ on the boundary, then $\langle a, e_i\rangle = 0$ for every $1 \le i \le n-1$, and $a^{\mathrm{T}} = \langle \nu, a\rangle\nu$ on ∂M. On the other hand, let η be the outward pointing unitary normal to Σ in Π. Observe that $\eta = e_1 \times \cdots \times e_{n-1} \times a$, and similarly $\nu = e_1 \times \cdots \times e_{n-1} \times N$, with the conditionals $\det(\eta, e_1 \ldots, e_{n-1}, a) = 1$ and $\det(\nu, e_1 \ldots, e_{n-1}, N) = 1$. Hence, we compute

$$\begin{aligned}\langle \nu, \eta\rangle &= \det(\nu, e_1, \ldots, e_{n-1}, a) = \det(\nu, e_1, \ldots, e_{n-1}, \langle \nu, a\rangle\nu - \langle a, N\rangle N)\\ &= -\langle a, N\rangle \det(\nu, e_1, \ldots, e_{n-1}, N) = -\langle a, N\rangle,\end{aligned}$$

and, similarly,

$$\langle N, \eta\rangle = -\langle \nu, a\rangle.$$

Let A_Σ denote the shape operator of Σ (as an $(n-1)$-dimensional hypersurface of the Euclidean space $\Pi \equiv \mathbb{E}^n$) with respect to the normal η. Note that the inclusion $\Pi \subset \mathbb{L}^{n+1}$ is totally geodesic, and so we have

$$\nabla^0_{e_j} e_i = \sum_{k=1}^{n-1} \langle \nabla^0_{e_j} e_i, e_k\rangle e_k + \langle \nabla^0_{e_j} e_i, \nu\rangle\nu - \langle A(e_i), e_j\rangle N, \quad \text{for every } 1 \le i, j \le n-1$$

and also

$$\nabla^0_{e_j} e_i = \sum_{k=1}^{n-1} \langle \nabla^0_{e_j} e_i, e_k\rangle e_k + \langle A_\Sigma(e_i), e_j\rangle \eta,$$

so that

$$\langle Ae_i, e_j\rangle = \langle A_\Sigma e_i, e_j\rangle\langle \eta, N\rangle = -\langle A_\Sigma e_i, e_j\rangle\langle \nu, a\rangle,$$

since we have already observed that $\langle \eta, N\rangle = -\langle \nu, a\rangle$.

We now suppose that the basis $\{e_1, \ldots, e_{n-1}\}$ on the boundary ∂M is chosen such that it is formed by the eigenvectors of A_Σ, with eigenvalues given by τ_i, that is

$$A_\Sigma e_i = \tau_i e_i, \quad 1 \le i \le n-1.$$

Hence, $\langle Ae_i, e_j\rangle = 0$ when $i \ne j$, and for each $p \in \partial M$, the matrix of A in the orthonormal basis $\{e_1, \ldots, e_{n-1}, \nu\}$ of T_pM is given by

$$A = \begin{pmatrix} -\tau_1\langle \nu, a\rangle & 0 & \cdots & 0 & \langle A\nu, e_1\rangle \\ 0 & -\tau_2\langle \nu, a\rangle & \cdots & 0 & \langle A\nu, e_2\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\tau_{n-1}\langle \nu, a\rangle & \langle A\nu, e_{n-1}\rangle \\ \langle A\nu, e_1\rangle & \langle A\nu, e_2\rangle & \cdots & \langle A\nu, e_{n-1}\rangle & \langle A\nu, \nu\rangle \end{pmatrix}.$$

This expression invites us to compute the characteristic polynomial of A. To do that, we begin by observing that the following recursive formula holds:

$$\Delta_k = (t + \tau_k \langle \nu, a\rangle)\Delta_{k-1} - \langle A\nu, e_k\rangle^2 \prod_{j=1}^{k-1}(t + \tau_j \langle \nu, a\rangle), \quad \text{for every } k, \quad 1 \le k \le n-1,$$

where $\Delta_k = \det(tI_{k+1} - \Lambda_k)$ denotes the characteristic polynomial of the $(k+1)$-dimensional matrix

$$\Lambda_k = \begin{pmatrix} -\tau_1\langle \nu, a\rangle & 0 & \cdots & 0 & \langle A\nu, e_1\rangle \\ 0 & -\tau_2\langle \nu, a\rangle & \cdots & 0 & \langle A\nu, e_2\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\tau_k\langle \nu, a\rangle & \langle A\nu, e_k\rangle \\ \langle A\nu, e_1\rangle & \langle A\nu, e_2\rangle & \cdots & \langle A\nu, e_k\rangle & \langle A\nu, \nu\rangle \end{pmatrix}.$$

In particular, $\Delta_{n-1} = \det(tI_n - A)$ is the characteristic polynomial of A, and therefore, applying a simple induction argument, we obtain

$$\begin{aligned}\det(tI_n - A) &= (t - \langle A\nu, \nu\rangle)\prod_{i=1}^{n-1}(t + \tau_i\langle \nu, a\rangle) - \sum_{i=1}^{n-1}\langle A\nu, e_i\rangle^2 \prod_{j=1, j\neq i}^{n-1}(t + \tau_j\langle \nu, a\rangle) \\ &= (t - \langle A\nu, \nu\rangle)\sum_{i=0}^{n-1} s_i\langle \nu, a\rangle^i t^{n-1-i} - \sum_{i=1}^{n-1}\langle A\nu, e_i\rangle^2 \sum_{j=0}^{n-2} s_j(\hat{\tau}_i)\langle \nu, a\rangle^j t^{n-2-j},\end{aligned}$$

where s_r ($s_r(\hat{\tau}_i)$, respectively) are the symmetric functions of $\tau_1, \dots, \tau_{n-1}$, ($\tau_1, \dots, \hat{\tau}_i, \dots, \tau_{n-1}$, respectively), and, as usual, $s_0 = 1$ by definition. Comparing the terms of above polynomials, we conclude from Eq. (5) that

$$\begin{aligned}\binom{n}{r} H_r = (-1)^r S_r &= s_r\langle \nu, a\rangle^r - s_{r-1}\langle \nu, a\rangle^{r-1}\langle A\nu, \nu\rangle \\ &\quad -\langle \nu, a\rangle^{r-2}\sum_{i=1}^{n-1} s_{r-2}(\hat{\tau}_i)\langle A\nu, e_i\rangle^2, \qquad (16)\end{aligned}$$

where $s_{-1} = s_n = 0$ by definition. Now, we are ready to state and prove the following essential auxiliary result.

Lemma 2. *Let $x : M \to \mathbb{L}^{n+1}$ be a spacelike hypersurface with hyperplanar boundary $\Sigma = x(\partial M)$, and assume that Σ is contained in a hyperplane $\Pi = a^{\perp}$, 'a' being the unit timelike vector which is the same time-orientation as N. Then*

$$\langle T_r\nu, a\rangle = s_r\langle \nu, a\rangle^{r+1}, \quad \textit{for every } 0 \le r \le n-1, \qquad (17)$$

where s_r are the symmetric functions of the eigenvalues of A_Σ, the shape operator of Σ in Π with respect to the outward pointing unitary normal.

Proof. We first note that Eq. (17) trivially holds for $r = 0$. Besides, observe that

$$\langle T_r\nu, a\rangle = \langle T_r\nu, a^{\mathrm{T}}\rangle = \langle \nu, a\rangle\langle T_r\nu, \nu\rangle. \tag{18}$$

Hence, it suffices to compute $\langle T_r\nu, \nu\rangle$ when $r \geq 1$. For this, we use induction on r. For $r = 1$, Eq. (16) becomes $nH_1 = s_1\langle \nu, a\rangle - \langle A\nu, \nu\rangle$, which implies $\langle T_1\nu, \nu\rangle = s_1\langle \nu, a\rangle$. Suppose that

$$\langle T_j\nu, \nu\rangle = s_j\langle \nu, a\rangle^j, \quad \text{for all } j, \quad 1 \leq j < r. \tag{19}$$

Therefore, from the inductive definition of T_r in Eqs. (4) and (19), we have

$$\begin{aligned}\langle T_r\nu, \nu\rangle &= \binom{n}{r} H_r + \langle T_{r-1}\nu, A\nu\rangle \\ &= \binom{n}{r} H_r + \sum_{i=1}^{n-1}\langle A\nu, e_i\rangle\langle T_{r-1}\nu, e_i\rangle + \langle A\nu, \nu\rangle\langle T_{r-1}\nu, \nu\rangle \\ &= \binom{n}{r} H_r + \sum_{i=1}^{n-1}\langle A\nu, e_i\rangle\langle T_{r-1}\nu, e_i\rangle + s_{r-1}\langle \nu, a\rangle^{r-1}\langle A\nu, \nu\rangle.\end{aligned} \tag{20}$$

On the other hand, we also know that

$$Ae_i = -\tau_i\langle \nu, a\rangle e_i + \langle A\nu, e_i\rangle\nu,$$

so that from our induction hypothesis Eqs. (4) and (19), we have

$$\langle T_j\nu, e_i\rangle = \langle T_{j-1}\nu, Ae_i\rangle = -\tau_i\langle \nu, a\rangle\langle T_{j-1}\nu, e_i\rangle + s_{j-1}\langle \nu, a\rangle^{j-1}\langle A\nu, e_i\rangle,$$
$$\text{for every } 0 \leq j \leq r.$$

This implies by a recursive argument that

$$\langle T_{r-1}\nu, e_i\rangle = \left(\sum_{j=0}^{r-2}(-1)^j\tau_i^j s_{r-2-j}\right)\langle \nu, a\rangle^{r-2}\langle A\nu, e_i\rangle. \tag{21}$$

Now, it is not difficult to see that

$$s_m(\hat{\tau}_i) = \sum_{j=0}^{m}(-1)^j\tau_i^j s_{m-j}, \quad \text{for every } 1 \leq m \leq n-1,$$

so that Eq. (21) becomes

$$\langle T_{r-1}\nu, e_i\rangle = s_{r-2}(\hat{\tau}_i)\langle \nu, a\rangle^{r-2}\langle A\nu, e_i\rangle.$$

Using this in Eq. (20), together with the expression for $\binom{n}{r} H_r$ given in Eq. (16), we conclude that

$$\langle T_r\nu, \nu\rangle = s_r\langle \nu, a\rangle^r,$$

which jointly with Eq. (18) gives the desired formula (17). This finishes the proof of Lemma 2. □

Let us denote h_r by the rth mean curvature of the boundary Σ as an Euclidean $(n-1)$-dimensional hypersurface of $\Pi \equiv \mathbb{E}^n$. That is

$$\binom{n-1}{r} h_r = s_r, \quad \text{for every } 1 \leq r \leq n-1.$$

Then, using Lemma 2 in formula (11) with $Y = a$, we obtain that

$$\oint_{\partial M} h_{r-1} \langle \nu, a \rangle^r \, \mathrm{d}s = n \int_M H_r \langle a, N \rangle \, \mathrm{d}M, \quad \text{for every } 1 \leq r \leq n, \tag{22}$$

since $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$. In particular, if the rth mean curvature H_r is constant, Lemma 2 allows us to rewrite our flux formula (15) as

$$\oint_{\partial M} h_{r-1} \langle \nu, a \rangle^r \, \mathrm{d}s = -nH_r \operatorname{vol}(D). \tag{23}$$

Remark 1. It is interesting to remark that, in contrast to the Euclidean case, Eq. (23) does not imply here any restriction on the possible values of the constant rth mean curvature. For instance, in [4], it is shown that if $\Sigma = \mathbb{S}^{n-1}$ is an $(n-1)$-dimensional sphere of radius one and M is an immersed compact hypersurface in the Euclidean space bounded by $\mathbb{S}^{n-1}$ whose rth mean curvature H_r is constant, then we have that $0 \leq |H_r| \leq 1$. However, in the case of the Minkowski space–time, the family of hyperbolic caps

$$M_\lambda = \{x \in \mathbb{L}^{n+1} : \langle x, x \rangle = -\lambda^2, 0 < x_0 \leq \sqrt{1+\lambda^2}\}, \quad \text{with } 0 < \lambda < \infty,$$

describes a family of spacelike compact hypersurfaces in $\mathbb{L}^{n+1}$ bounded by $\mathbb{S}^{n-1}$ with constant rth mean curvature $H_r(\lambda) = 1/\lambda^r$.

5. Hypersurfaces with constant intrinsic higher order mean curvature

We are now in a position to prove our Theorem 1. Suppose that the rth mean curvature H_r is constant, where r is even, and that the boundary $\Sigma = x(\partial M)$ is a round sphere $\mathbb{S}^{n-1}(\rho)$ of radius $\rho > 0$. In that case, we have that $\tau_i = -1/\rho$, for every $i = 1, \ldots, n-1$, so that $h_{r-1} = -1/\rho^{r-1}$. Besides, $\operatorname{vol}(D) = \rho A_\rho / n$, where $A_\rho = \operatorname{area}(\mathbb{S}^{n-1}(\rho))$. Then our flux formula (23) becomes

$$\oint_{\partial M} \langle \nu, a \rangle^r \, \mathrm{d}s = \rho^r H_r A_\rho. \tag{24}$$

We may assume without loss of generality that the constant H_r is positive. Otherwise, we know from Corollary 1 that the hypersurface is a hyperplanar ball. By the Holder inequality, we obtain from Eq. (24) that

$$\left|\oint_{\partial M}\langle\nu,a\rangle\,\mathrm{d}s\right| \le \left(\oint_{\partial M}\langle\nu,a\rangle^r\,\mathrm{d}s\right)^{1/r} A_\rho^{(r-1)/r} = \rho H_r^{1/r}A_\rho. \tag{25}$$

On the other hand, we know also know by Corollary 1 that

$$H_1 \ge H_r^{1/r} > 0,$$

with equality only at umbilical points. Therefore, we have

$$nH_1(-\langle a,N\rangle) \ge nH_r^{1/r}(-\langle a,N\rangle) > 0, \tag{26}$$

with equality if and only if M is totally umbilical. Besides, we also know from Eq. (22) that

$$\oint_{\partial M}\langle\nu,a\rangle\,\mathrm{d}s = n\int_M H_1\langle a,N\rangle\,\mathrm{d}M < 0.$$

Then, integrating Eq. (26) on M and using Eq. (14), we deduce that

$$\begin{aligned}\left|\oint_{\partial M}\langle\nu,a\rangle\,\mathrm{d}s\right| &= n\int_M H_1(-\langle a,N\rangle)\,\mathrm{d}M \ge nH_r^{1/r}\int_M(-\langle a,N\rangle)\,\mathrm{d}M\\ &= nH_r^{1/r}\,\mathrm{vol}(D) = \rho H_r^{1/r}A_\rho,\end{aligned} \tag{27}$$

with equality if and only if M is totally umbilical. Finally, from Eq. (25), we have the equality in Eq. (27) and then M must be umbilical. This finishes the proof of our result.

6. An integral inequality

In this section, we will derive an integral inequality which, jointly with our flux formula in Proposition 1, will allow us to prove our result for the case of extrinsic higher order mean curvatures. To do that, observe that from Eq. (10) with $r = 0$, we know that

$$\mathrm{div}_M(Y^{\mathrm{T}}) = nH_1\langle Y,N\rangle,$$

for a Killing vector field Y on $\mathbb{L}^{n+1}$, and also

$$\mathrm{div}_M(T_rY^{\mathrm{T}}) = (r+1)\binom{n}{r+1}H_{r+1}\langle Y,N\rangle.$$

Therefore, if H_r is constant, we conclude that

$$\mathrm{div}_M\left(T_rY^{\mathrm{T}} - \frac{n-r}{n}\binom{n}{r}H_rY^{\mathrm{T}}\right) = -(n-r)\binom{n}{r}(H_1H_r - H_{r+1})\langle Y,N\rangle. \tag{28}$$

This equation is the key for the proof of the following result, which generalizes and integral inequality given in [5] (Proposition 3). It is worth pointing out that Eq. (28) is meaningful when $r \leq n-1$, since for $r = n$, we know that $T_n = 0$ and both sides of Eq. (28) trivially vanish.

Proposition 2. *Let $x : M \to \mathbb{L}^{n+1}$ be a spacelike immersion of a compact hypersurface bounded by an $(n-1)$-dimensional embedded submanifold $\Sigma = x(\partial M)$, and assume that Σ is contained in a hyperplane Π* of *$\mathbb{L}^{n+1}$. Let 'a' be the unit timelike vector in $\mathbb{L}^{n+1}$ such that $\Pi = a^{\perp}$ which is the same time-orientation as N. If the 'r'th mean curvatureH_r is constant, with $1 \leq r \leq n-1$, then*

$$\oint_{\partial M} h_r \langle \nu, a\rangle^{r+1}\,\mathrm{d}s \geq H_r \oint_{\partial M} \langle \nu, a\rangle\,\mathrm{d}s, \tag{29}$$

where h_r stands for the 'r'th mean curvature of Σ in Π with respect to the outward pointing unitary normal. Moreover, the equality holds if and only if M is totally umbilical.

Proof. We may assume without loss of generality that the constant H_r is positive. Otherwise, we know from Corollary 1 that the hypersurface is itself hyperplanar and Eq. (29) trivially holds since $\langle \nu, a\rangle = 0$ on ∂M.

Choosing the constant Killing field $Y = a$ in Eq. (28) and integrating on M, we obtain by the divergence theorem that

$$\begin{aligned} &-(n-r)\binom{n}{r}\int_M (H_1 H_r - H_{r+1})\langle a, N\rangle\,\mathrm{d}M \\ &\quad = \oint_{\partial M} \langle T_r \nu, a\rangle\,\mathrm{d}s - \frac{n-r}{n}\binom{n}{r} H_r \oint_{\partial M} \langle \nu, a\rangle\,\mathrm{d}s. \end{aligned} \tag{30}$$

Since N and a are in the same time-cone, then $\langle a, N\rangle \leq -1 < 0$. In the case, where $r = 1$, the term $H_1 H_r - H_{r+1}$ reduces to $H_1^2 - H_2$, which by Cauchy–Schwarz inequality clearly satisfies $H_1^2 - H_2 \geq 0$, equality holding at umbilical points. For the general case, we know from Corollary 1 that

$$H_{r-1} \geq H_r^{(r-1)/r} > 0,$$

and also

$$H_1 \geq H_{r-1}^{1/(r-1)},$$

with equality only at umbilical points. On the other hand, it is known ([13], Theorem 55) that $H_r^2 - H_{r-1}H_{r+1} \geq 0$, equality holding at umbilical points, so that

$$H_{r+1} \leq \frac{H_r^2}{H_{r-1}}.$$

Therefore

$$\begin{aligned} H_1 H_r - H_{r+1} &\geq \frac{H_r}{H_{r-1}}(H_1 H_{r-1} - H_r) \geq \frac{H_r}{H_{r-1}}(H_1 H_{r-1} - H_r^{r/(r-1)}) \\ &= H_r(H_1 - H_{r-1}^{1/(r-1)}) \geq 0, \end{aligned}$$

with equality at the umbilical points of the hypersurface. Therefore, from Eq. (30), we conclude that

$$\oint_{\partial M} \langle T_r \nu, a\rangle \, \mathrm{d}s \geq \frac{n-r}{n} \binom{n}{r} H_r \oint_{\partial M} \langle \nu, a\rangle \, \mathrm{d}s, \tag{31}$$

with equality if and only if the hypersurface is totally umbilical. Finally, using Lemma 2, Eq. (31) becomes Eq. (29). □

7. Hypersurfaces with constant extrinsic higher order mean curvature

We are now ready to prove Theorem 2. Suppose that the rth mean curvature H_r is constant, where r is now odd, and that the boundary $\Sigma = x(\partial M)$ is a round sphere $\mathbb{S}^{n-1}(\rho)$ of radius $\rho > 0$. In that case, we have that $\tau_i = -1/\rho$, for every $i = 1, \dots, n-1$, so that $h_r = -1/\rho^r$.

Proof of Theorem 2 (When $1 \leq r < n$). If $r < n$, then we may apply Proposition 2 and the inequality Eq. (29) becomes

$$\oint_{\partial M} \langle \nu, a\rangle^{r+1} \, \mathrm{d}s \leq -\rho^r H_r \oint_{\partial M} \langle \nu, a\rangle \, \mathrm{d}s = \rho^r H_r \left| \oint_{\partial M} \langle \nu, a\rangle \, \mathrm{d}s \right|, \tag{32}$$

since we also know from Eq. (22) that

$$\oint_{\partial M} \langle \nu, a\rangle \, \mathrm{d}s = n \int_M H_1 \langle a, N\rangle \, \mathrm{d}M < 0. \tag{33}$$

Indeed, we may assume without loss of generality, as in the proof of Theorem 1, that the constant H_r is positive, and

$$H_1 \geq H_r^{1/r} > 0. \tag{34}$$

Otherwise, we know from Corollary 1 that the hypersurface is a hyperplanar ball. Besides, Eq. (32) becomes an equality if and only if M is totally umbilical.

On the other hand, by the Holder inequality, we obtain that

$$\left| \oint_{\partial M} \langle \nu, a\rangle \, \mathrm{d}s \right| \leq \left(\oint_{\partial M} \langle \nu, a\rangle^{r+1} \, \mathrm{d}s \right)^{1/(r+1)} A_\rho^{r/(r+1)},$$

or, equivalently,

$$\oint_{\partial M} \langle \nu, a\rangle^{r+1} \, \mathrm{d}s \geq \frac{1}{A_\rho^r} \left| \oint_{\partial M} \langle \nu, a\rangle \, \mathrm{d}s \right|^{r+1}. \tag{35}$$

Moreover, from Eqs. (33) and (34), also using Eq. (14), we get

$$\left| \oint_{\partial M} \langle \nu, a\rangle \, \mathrm{d}s \right| \geq n H_r^{1/r} \int_M (-\langle a, N\rangle) \, \mathrm{d}M = n H_r^{1/r} \operatorname{vol}(D) = \rho H_r^{1/r} A_\rho, \tag{36}$$

which jointly with Eq. (35) implies

$$\oint_{\partial M} \langle \nu, a\rangle^{r+1}\,\mathrm{d}s \geq \rho^r H_r \left|\oint_{\partial M} \langle \nu, a\rangle\,\mathrm{d}s\right|.$$

This means that we have the equality in Eq. (32) and then M must be umbilical. This finishes the proof of our result when $r < n$. □

Observe that this proof does not work $r = n$ is odd. For that case, we will provide another proof which uses the ideas of our proof of Theorem 1. In fact, this new proof works not only when $r = n$ is odd but also when $r > 1$ is odd.

Proof of Theorem 2 (When $1 < r \leq n$). As usual, we may assume that the constant H_r is positive. Since r is odd, $h_{r-1} = 1/\rho^{r-1}$ and our flux formula (23) becomes

$$\oint_{\partial M} \langle \nu, a\rangle^{r}\,\mathrm{d}s = -\rho^r H_r A_\rho < 0. \tag{37}$$

Now, the key of the proof is to realize that the function $\langle \nu, a\rangle$ does not vanish on ∂M. This follows from Eq. (16). In fact, if there exists a point $p_0 \in \partial M$, where $\langle \nu, a\rangle(p_0) = 0$, then from Eq. (16), the constant $H_r = H_r(p_0) = 0$ because $r \geq 3$, which cannot be possible. Therefore, the function $\langle \nu, a\rangle$ does not vanish on ∂M, and from Eq. (37), it is necessarily negative on ∂M. Therefore, $|\langle \nu, a\rangle|^r = -\langle \nu, a\rangle^r$, and by the Holder inequality, we also know from Eq. (37) that

$$\left|\oint_{\partial M} \langle \nu, a\rangle\,\mathrm{d}s\right| \leq \left(-\oint_{\partial M} \langle \nu, a\rangle^r\,\mathrm{d}s\right)^{1/r} A_\rho^{(r-1)/r} = \rho H_r^{1/r} A_\rho.$$

From here, the proof works as the proof of Theorem 2. □

Acknowledgements

This work was done while the first author was visiting the Institute des Hautes Études Scientifiques at Bures-sur-Yvette, France. He would like to thank that institution for their wonderful hospitality. L.J. Alías was partially supported by DGICYT Grant no. PB97-0784-C03-02 and Fundación Séneca (PRIDTYC), Consejería de Educación y Cultura CARM. J.M. Malacarne was partially supported by PICD/CAPES, Brazil.

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